Series Solution of Equations for Re-Entry Vehicles with Variable Lift and Drag Coefficients

Y. C. Shen*
Northrop Corporation, Hawthorne, Calif.

The complete two-dimensional equations of motion governing a re-entry vehicle considered as a point mass in a nonrotating exponential atmosphere, with lift and drag coefficients given arbitrarily as functions of both velocity and altitude, are solved by using series expansions. The only assumption made in this solution is that the variation of altitude relative to the earth or planetary radius is small. For constant lift and drag, the solution not only checks with exact numerical result, shows agreement with the existing approximate solutions, but also serves as an estimate of order of accuracy for such solutions. For variable lift and drag, the result checks excellently with the exact numerical solution.

Nomenclature

$\bar{A}$	=	reference area of the vehicle
C_D	=	drag coefficient of the vehicle
C_L	=	lift coefficient of the vehicle
$\bar{G}$	=	universal gravitational constant
$\bar{M}$	=	mass of the earth or planet
$\bar{m}$	=	mass of the vehicle
$\bar{R}$	=	radius of the earth or planet
$\bar{R} + \bar{r}$	=	radial distance from center of earth or planet to vehicle
$\bar{t}$	=	time
$\bar{\rho}$	=	atmospheric density
θ	=	angle between flight path and the horizontal

Introduction

RECENTLY, there have been numerous publications[1–18] concerning approximate solutions for re-entry trajectories for both nonlifting and lifting vehicles with constant aerodynamic coefficients. As a result, considerable understanding about the dynamic and thermodynamic behavior of re-entry vehicles with constant lift and drag coefficients has been gained. However, for a re-entry vehicle with lift and drag coefficients given arbitrarily as functions of velocity and altitude, analytical solutions are meager. Wang and Chu[19] studied the case of variable lifting, where the flight path angle is small, and the lift coefficient is represented by $C_L = C_{LE} - \zeta X^n$(C_{LE} is the lift coefficient at entry, ζ and n are the two lift parameters to be varied to approximate most practical lift programs, and X is proportional to atmospheric density). They attacked the problem by assuming a solution for the flight path angle θ in a series form

$$\theta = \Sigma a_{ij} X^i (\log X/X_E)^j$$

Although their solution suggests a way of achieving a smooth transition from an initial plunge into a nominal glide phase, its applicability is somewhat limited as a consequence of the following reasons:

1) The flight path angle is small.

2) The lift coefficient assumes a special form and varies with altitude only.

3) There is difficulty in obtaining a general solution with respect to lift parameter n.

4) There is inconvenience in adjusting ζ and n to approximate a practical lift program.

5) The strange variation of drag with lift of re-entry type vehicle may deviate considerably from the parabolic lift-drag polar.

With a view to removing these restrictions and providing a single solution for re-entry vehicle with lift and drag coefficients given arbitrarily as functions of both altitude and velocity, the present analysis is made.

Equations of Motion

The use of a two-dimensional inertial coordinate system with its origin at the center of a spherical, nonrotating earth or planet is shown in Fig. 1. The two-dimensional equations of motion for a lifting vehicle, considered as a point mass, flying in the stationary earth or planetary atmosphere, can be written as

$$d\bar{v}/d\bar{t} = -(\tfrac{1}{2}\bar{m})C_D(\bar{v},\bar{r})\bar{\rho}\bar{v}^2\bar{A} + [\bar{G}\,\bar{M}/(\bar{R}+\bar{r})^2]\sin\theta \quad (1a)$$

$$-\bar{v}(d\theta/d)\bar{t} = (\tfrac{1}{2}\bar{m})C_L(\bar{v},\bar{r})\bar{\rho}\bar{v}^2\bar{A} - [\bar{G}\bar{M}/(\bar{R}+\bar{r})^2 - \bar{v}^2/(\bar{R}+\bar{r})]\cos\theta \quad (1b)$$

If the following dimensionless quantities are chosen:

$$r = \bar{r}/\bar{R} \qquad t = \bar{t}/\bar{T} \qquad v = \bar{T}\bar{v}/\bar{R}\dagger \qquad \rho = \bar{A}\bar{R}\bar{\rho}/\bar{m}$$

where

$$\bar{T} = (\bar{R}^3/\bar{G}\bar{M})^{1/2}$$

Eqs. (1a) and (1b) may be rewritten in the nondimensional form:

$$dv/dt = -(\tfrac{1}{2})C_D(v,r)\rho v^2 + \sin\theta/(1+r)^2 \quad (2a)$$

$$-v d\theta/dt = (\tfrac{1}{2})C_L(v,r)\rho v^2 - \cos\theta/(1+r)^2 + v^2\cos\theta/(1+r) \quad (2b)$$

Introducing the kinematic relation

$$dr/dt = -v\sin\theta \quad (3)$$

into Eqs. (2a) and (2b) to change the independent variable from t to r, the following results:

$$dv^2/dr - C_D(v,r)\rho v^2/\sin\theta + 2/(1+r)^2 = 0 \quad (4a)$$

$$d\cos\theta/dr + \cos\theta/(1+r) - \cos\theta/(1+r)^2v^2 + (\tfrac{1}{2})C_L(v,r)\rho = 0 \quad (4b)$$

Equations (4a) and (4b) are the basic unapproximated equations governing the planar motion of a point mass vehicle in the stationary atmosphere of a spherical, nonrotating earth, or planet.

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* Senior Scientist; presently with Hughes Aircraft Company, El Segundo, Calif.

† Notice that the dimensionless v is actually the ratio between vehicle velocity $\bar{v}$ and circular velocity $\bar{v}_c$, since $\bar{R}/\bar{T} = (\bar{G}M/\bar{R})^{1/2} = \bar{v}_c$.

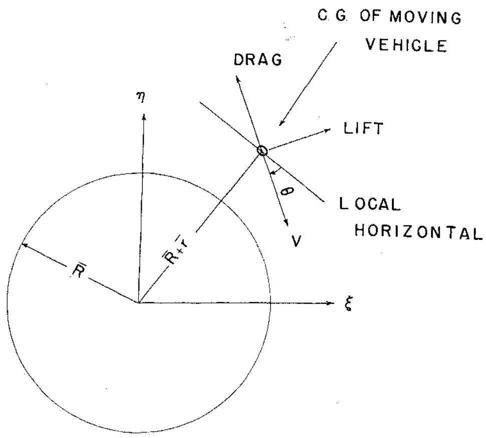

Fig. 1 Inertial coordinate system.

However, since lifting entry occurs at low altitude, therefore r is very small, Eqs. (4a) and (4b) can thus be simplified as

$$dv^2/dr - C_D(v,r)\rho v^2/\sin\theta + 2 = 0 \tag{5a}$$

$$d\cos\theta/dr + (1 - 1/v^2)\cos\theta + (\tfrac{1}{2})C_L(v,r)\rho = 0 \tag{5b}$$

For exponential earth or planetary atmosphere, r is related to ρ as

$$\rho = \rho_0 \exp(-\bar{\beta}\bar{R}r) \tag{6}$$

where ρ_0 is the dimensionless density of the earth or planetary atmosphere at the surface of the earth or planet, i.e., at $r = 0$.

By means of Eq. (6), Eqs. (5a) and (5b) can be transformed, with ρ as independent variable, as follows:

$$dv^2/d\log\rho - \epsilon[2 - C_D(v,\rho)\rho v^2/\sin\theta] = 0 \tag{7a}$$

$$d\cos\theta/d\log\rho - \epsilon[(1 - 1/v^2)\cos\theta + C_L(v,\rho)\rho/2] = 0 \tag{7b}$$

Where $\epsilon = 1/\bar{\beta}\bar{R}$, Eqs. (7a) and (7b) can also be rewritten as

$$(1 - \cos^2\theta)[(dv^2/d\log\rho) - (2\epsilon)]^2 = \epsilon^2[C_D(v,\rho)\rho v^2]^2 \tag{8a}$$

$$v^2[(d\cos\theta/d\log\rho) - \epsilon\cos\theta - \epsilon C_L(v,\rho)\rho/2] = -\epsilon\cos\theta \tag{8b}$$

Equations (7a) and (7b), as will be seen immediately, provide considerable information about re-entry trajectory. However, for actual manipulation, Eqs. (8a) and (8b) are used since they are in a form more suitable for series solution.

Equations (8a) and (8b) are to be solved with the initial conditions

$$v(\rho_e) = v_e \tag{9a}$$

$$\cos\theta(\rho_e) = \cos\theta_e \tag{9b}$$

where $(\ )_e$ denotes quantity at entry.

Significance of Eqs. (7a) and (7b) and Justification for Assumptions in Existing Solutions

The following are worthy of note:

1) Equations (7a) and (7b) are independent of $\bar{A}/\bar{m}$; thus the solution of them, independent of $\bar{A}/\bar{m}$, may be called similarity solution (a term that was used previously[20] for trajectory calculation).

2) For small ϵ, and at high altitude, $(dv^2/d\log\rho) \approx 0$, $(d\cos\theta/d\log\rho) \approx 0$. This appears to be the mathematical justification of the simplifications used in the various existing approximate solutions to replace $\cos\theta$ by $\cos\theta_e$, $\sin\theta$ by $\sin\theta_e$, and v by v_e.

3) It can easily be shown that, in Eq. (7a), the gravitational force is generally (but not always) smaller than the drag. In Eq. (7b), the difference between gravitational and centrifugal force is generally (but not always) smaller than lift. Thus, they can be neglected in most cases as was done in some of the existing solutions.

4) Since $\epsilon(1 - 1/v^2)\cos\theta$ is generally small, it is thus insensitive to the integration. This insensitivity constitutes the basis of analysis in Ref. 17.

Series Expansion

Since ϵ is a small parameter in the two simultaneous differential equations [Eqs. (7a) and (7b)], this naturally suggests series solution in the following form:

$$v = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \epsilon^3 f^{(3)} + \ldots \tag{10a}$$

$$\cos\theta = h^{(0)} + \epsilon h^{(1)} + \epsilon^2 h^{(2)} + \epsilon^3 h^{(3)} + \ldots \tag{10b}$$

where, because of Eqs. (9a) and (9b), the $f^{(n)}$ and $h^{(n)}$ satisfy the initial conditions

$$f^{(0)}(\rho_e) = v_e \qquad f^{(n)}(\rho_e) = 0 \qquad n \neq 0 \tag{11a}$$

$$h^{(0)}(\rho_e) = \cos\theta_e \qquad h^{(n)}(\rho_e) = 0 \qquad n \neq 0 \tag{11b}$$

Furthermore, because of the series expansion, the aerodynamic coefficients can be expanded as follows:

$$\begin{aligned} C_D(v,\rho) &= C_D[f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \ldots, \rho] \\ &= C_D(f^{(0)},\rho) + (\partial C_D/\partial v)(f^{(0)},\rho)\,[\epsilon f^{(1)} + \epsilon^2 f^{(2)} + \ldots] + \tfrac{1}{2}(\partial^2 C_D/\partial v^2)(f^{(0)},\rho)\,[\epsilon f^{(1)} + \epsilon^2 f^{(2)} + \ldots]^2 + \tfrac{1}{6}(\partial^3 C_D/\partial v^3)(f^{(0)},\rho)\,[\epsilon f^{(1)} + \epsilon^2 f^{(2)} + \ldots]^3 + \ldots \\ &= C_D(f^{(0)},\rho) + \epsilon(\partial C_D/\partial v)(f^{(0)},\rho)f^{(1)} + \epsilon^2[(\partial C_D/\partial v)(f^{(0)},\rho)f^{(2)} + \tfrac{1}{2}(\partial^2 C_D/\partial v^2)(f^{(0)},\rho)f^{(1)2}] + \epsilon^3[(\partial C_D/\partial v)(f^{(0)},\rho)f^{(3)} + (\partial^2 C_D/\partial v^2)(f^{(0)},\rho)f^{(1)}f^{(2)} + \tfrac{1}{6}(\partial^3 C_D/\partial v^3)(f^{(0)},\rho)f^{(1)3}] + \ldots \end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned} C_L(v,\rho) = {} & C_L(f^{(0)},\rho) + \epsilon(\partial C_L/\partial v)(f^{(0)},\rho)f^{(1)} + \epsilon^2[(\partial C_L/\partial v)(f^{(0)},\rho)f^{(2)} + \tfrac{1}{2}(\partial^2 C_D/\partial v^2)(f^{(0)},\rho)f^{(1)2}] + \\ & \epsilon^3[(\partial C_L/\partial v)(f^{(0)},\rho)f^{(3)} + (\partial^2 C_D/\partial v^2)(f^{(0)},\rho)f^{(1)}f^{(2)} + \tfrac{1}{6}(\partial^3 C_D/\partial v^3)(f^{(0)},\rho)f^{(1)3}] + \ldots \end{aligned} \tag{13}$$

Substituting Eqs. (10a, 10b, 12, and 13) into Eqs. (8a) and (8b), and equating coefficients for zeroth power of ϵ, one has

$$(1 - h^{(0)2})\left(\frac{df^{(0)2}}{d\log\rho}\right)^2 = 0 \tag{14a}$$

$$f^{(0)2}\,\frac{dh^{(0)}}{d\log\rho} = 0 \tag{14b}$$

which, together with the initial conditions [Eqs. (11a) and (11b)] $f^{(0)}(\rho_e) = v_e$ and $h^{(0)}(\rho_e) = \cos\theta_e$, lead to zeroth order solution:

$$f^{(0)} = v_e \tag{15a}$$

$$h^{(0)} = \cos\theta_e \tag{15b}$$

The series solution will now assume the form

$$v = v_e + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \epsilon^3 f^{(3)} + \ldots \tag{16a}$$

$$\cos\theta = \cos\theta_e + \epsilon h^{(1)} + \epsilon^2 h^{(2)} + \epsilon^3 h^{(3)} + \ldots \tag{16b}$$

which, together with Eqs. (12) and (13), are again substituted into Eqs. (8a) and (8b). After equating coefficients of equal powers of ϵ and taking account of the decelerating effect of the drag force to determine the proper sign for the resulting differential equations, a slight manipulation will yield the following sequence.

First,

$$-2\sin\theta_e[v_e(df^{(1)}/d\log\rho) - 1] = C_D(v_e,\rho)\rho v_e^2 \tag{17a}$$

$$v_e^2[(dh^{(1)}/d\log\rho) - \cos\theta_e - C_L(v_e,\rho)\rho/2] = -\cos\theta_e \tag{17b}$$

Second,

$$-C_D(v_e,\rho)\rho v_e^2 \sin\theta_e[(df^{(1)2}/d\log\rho) + 2v_e(df^{(2)}/d\log\rho)] - (\cos\theta_e/\sin^2\theta_e)C_D^2(v_e,\rho)\rho^2 v_e^4 h^{(1)} = C_D(v_e,\rho)(\partial C_D/\partial v)(v_e,\rho)\rho^2 v_e^4 f^{(1)} + 2C_D^2(v_e,\rho)\rho^2 v_e^3 f^{(1)} \quad (18a)$$

$$v_e^2[(dh^{(2)}/d\log\rho) - h^{(1)} - (\rho/2)(\partial C_L/\partial v)(v_e,\rho)f^{(1)}] + 2v_e f^{(1)}[(dh^{(1)}/d\log\rho) - \cos\theta_e - (\rho/2)C_L(v_e,\rho)] = -h^{(1)} \quad (18b)$$

and so forth.

Equations (17a–18b) are first-order, linear differential equations in the form $dy/dx = g(x)$, which can be solved by direct integration where the integration constants are determined from Eqs. (11a) and (11b):

$$f^{(1)} = \left(\frac{1}{v_e}\right)\log\left(\frac{\rho}{\rho_e}\right) - \left(\frac{v_e}{2\sin\theta_e}\right)\int_{\rho_e}^{\rho} C_D(v_e,\rho)d\rho \quad (19a)$$

$$h^{(1)} = \left(1 - \frac{1}{v_e^2}\right)\cos\theta_e \log\left(\frac{\rho}{\rho_e}\right) + \left(\frac{1}{2}\right)\int_{\rho_e}^{\rho} C_L(v_e,\rho)d\rho \quad (19b)$$

$$f^{(2)} = -\left(\frac{1}{2v_e}\right)f^{(1)2} - \left(\frac{1}{2\sin\theta_e}\right)\int_{\rho_e}^{\rho}\left[\left(\frac{2}{v_e}\right)C_D(v_e,\rho) + \frac{\partial C_D}{\partial v}(v_e,\rho)\right]\log\left(\frac{\rho}{\rho_e}\right)d\rho + \left(\frac{v_e}{4\sin^2\theta_e}\right)\int_{\rho_e}^{\rho} d\rho \times \left[2C_D(v_e,\rho) + v_e\frac{\partial C_D}{\partial v}(v_e,\rho)\right]\int_{\rho_e}^{\rho} C_D(v_e,\rho)d\rho - \left(\frac{v_e\cos^2\theta_e}{2\sin^3\theta_e}\right)\left(1 - \frac{1}{v_e^2}\right)\int_{\rho_e}^{\rho} C_D(v_e,\rho)\log\left(\frac{\rho}{\rho_e}\right)d\rho - \frac{v_e\cos\theta_e}{4\sin^3\theta_e}\int_{\rho_e}^{\rho} d\rho\, C_D(v_e,\rho)\int_{\rho_e}^{\rho} C_L(v_e,\rho)d\rho \quad (20a)$$

$$h^{(2)} = \left(\frac{1}{2}\right)\left(1 - \frac{1}{v_e^2}\right)^2\cos\theta_e\left[\log\left(\frac{\rho}{\rho_e}\right)\right]^2 + \left(\frac{1}{2}\right)\left(1 - \frac{1}{v_e^2}\right)\int_{\rho_e}^{\rho}\frac{d\rho}{\rho}\int_{\rho_e}^{\rho} C_L(v_e,\rho)d\rho + \left(\frac{1}{2v_e}\right)\int_{\rho_e}^{\rho}\frac{\partial C_L}{\partial v}(v_e,\rho)\log\left(\frac{\rho}{\rho_e}\right)d\rho - \left(\frac{v_e}{4\sin\theta_e}\right)\int_{\rho_e}^{\rho} d\rho \times \frac{\partial C_L}{\partial v}(v_e,\rho)\int_{\rho_e}^{\rho} C_D(v_e,\rho)d\rho + \left(\frac{\cos\theta_e}{v_e^4}\right)\left[\log\left(\frac{\rho}{\rho_e}\right)\right]^2 - \left(\frac{\cos\theta_e}{v_e^2\sin\theta_e}\right)\int_{\rho_e}^{\rho}\frac{d\rho}{\rho}\int_{\rho_e}^{\rho} C_D(v_e,\rho)d\rho \quad (20b)$$

With $f^{(1)}$, $h^{(1)}$, $f^{(2)}$, and $h^{(2)}$ obtained as just shown, the re-entry trajectory is thus determined.

Comparison with Existing Approximate Solutions

As the velocity of the present series solution is expressed in a form quite different from that of the existing approximate solutions, it is difficult to make any direct comparison between them. However, the flight path angle of the present series solution is in a form that can be compared with that of Refs. 13 and 17.

The flight path angle in Ref. 17 can be expressed in terms of the notation of the present paper as follows:

$$\cos\theta = \cos\theta_e + \epsilon[(1 - 1/v^2)\cos\theta(\rho - \rho_e)/\rho + (\tfrac{1}{2})C_L(\rho - \rho_e)] \quad (21)$$

This agrees very well with the present series solution (for constant C_L and C_D):

$$\cos\theta = \cos\theta_e + \epsilon[(1 - 1/v_e^2)\cos\theta_e\log(\rho/\rho_e) + (\tfrac{1}{2})C_L(\rho - \rho_e)] \quad (22)$$

except that the term $(1 - 1/v^2)\cos\theta(\rho - \rho_e)/\rho$ in Eq. (21) is replaced by $(1 - 1/v_e^2)\cos\theta_e\log(\rho/\rho_e)$ in Eq. (22). Judging from this, the solution in Ref. 17 seems to be correct to the order $0(\epsilon^2)$.

In Ref. 13, the flight path angle, if written in terms of the present notation, is

$$\cos\theta = \cos\theta_e + \epsilon[(1 - 1/v_e^2)\cos\theta_e\log(\rho/\rho_e) + (\tfrac{1}{2})C_L(\rho - \rho_e)] - \epsilon^2(\cos\theta_e/v_e^2\sin\theta_e)(C_l/2) \times [C_D\rho_e\log(\rho/\rho_e) - (\rho - \rho_e)] + \ldots \quad (23)$$

This agrees well with the present series solution (for constant C_L and C_D):

$$\cos\theta = \cos\theta_e + \epsilon[(1 - 1/v_e^2)\cos\theta_e\log(\rho/\rho_e) + (\tfrac{1}{2})C_L(\rho - \rho_e)] - \epsilon^2\{[\tfrac{1}{2}(1 - 1/v_e^2)C_L - (\cos\theta_e/v_e^2\sin\theta_e)C_D]\rho_e\log(\rho/\rho_e) - [\tfrac{1}{2}(1 - 1/v_e^2)C_L - (\cos\theta_e/v_e^2\sin\theta_e)C_D] \times (\rho - \rho_e) - [\tfrac{1}{2}(1 - 1/v_e^2)^2 + 1/v_e^4] \times \cos\theta_e[\log(\rho/\rho_e)]^2\} + \ldots \quad (24)$$

The result of Ref. 13 appears to be accurate to the order of $0(\epsilon^3)$.

Comparison with Exact Numerical Solution

As a final check on the validity of the present series solutions, numerical examples are presented for the trajectory of

1) nonlifting vehicle with constant drag coefficient

$$C_D = 0.1 \qquad \bar{A}/\bar{m} = 0.25$$

2) lifting vehicle with constant lift and drag coefficient

$$C_L = 0.1 \qquad C_D = 0.41 \qquad \bar{A}/\bar{m} = 1$$

3) lifting vehicle with lift and drag coefficients varying with altitude alone

$$C_L = 0.1 + 1.29653 \times 10^{-3}(\rho - \rho_e) - 8.4049 \times 10^{-7}(\rho - \rho_e)^2 \ddagger$$

$$C_D = 0.4 + 0.6C_L^2 \qquad \bar{A}/\bar{m} = 0.5$$

4) lifting vehicle with lift and drag coefficients varying with both altitude and velocity

$$C_L = 0.1 + 1.29653 \times 10^{-3}(\rho - \rho_e) - 8.4049 \times 10^{-7}(\rho - \rho_e)^2 + (v - v_e) + (v - v_e)^2 \ddagger$$

$$C_D = 0.4 + 0.6C_L^2 \qquad \bar{A}/\bar{m} = 0.5$$

The results are in good agreement with the exact values from the numerical integration, as can be seen from Tables 1–4. Note that the velocity is not decreasing monotonously in this case as the gravitational force is dominating in part of the trajectory. This means that neglect of gravity term may sometimes cause error. In all the numerical examples given in the tables, the following constants are used:

$$1/\bar{\beta} = 24{,}000 \text{ ft}$$
$$\bar{R} = 2.0902 \times 10^7 \text{ ft}$$
$$\bar{\rho}_0 = 2.377 \times 10^{-3} \text{ slug/ft}^3$$

Conclusion

The equations of motion governing the planar entry of a vehicle with arbitrarily variable lift and drag have been reduced to a dimensionless form that not only provides physical and mathematical insight into re-entry trajectory but also suggests an approach for analytical solution by an expansion procedure in terms of the small parameter $(1/\bar{\beta}\bar{R})$.

‡ It should be noted that the C_L program given here may be unrealistic because C_L becomes prohibitively large at low altitude. However, at the re-entry portion of the trajectory, it is reasonable and does serve the purpose for numerical comparison between exact and series solutions.

Table 1 Nonlifting vehicle with constant drag coefficient

	Exact solution		Series solution	
$\bar{r}$, ft	θ, deg	$\bar{v}$, fps	θ, deg	$\bar{v}$, fps
300,000	5	30,000	5	30,000
274,398	4.79	30,025	4.79	30,026
249,891	4.57	30,045	4.58	30,047
226,490	4.36	30,056	4.38	30,058
204,208	4.15	30,044	4.17	30,048
183,065	3.93	29,979	3.98	29,989
163,093	3.73	29,798	3.83	29,826
144,345	3.54	29,375	3.80	29,458

Table 2 Lifting vehicle with constant lift and drag coefficient

	Exact solution		Series solution	
$\bar{r}$, ft	θ, deg	$\bar{v}$, fps	θ, deg	$\bar{v}$, fps
300,000	5	30,000	5	30,000
250,123	4.52	29,942	4.54	29,843
206,645	3.78	29,255	3.90	29,284
176,735	2.08	26,402	2.89	26,940

Table 3 Lifting vehicle with lift and drag coefficients varying with altitude alone

	Exact solution		Series solution	
$\bar{r}$, ft	θ, deg	$\bar{v}$, fps	θ, deg	$\bar{v}$, fps
300,000	5	30,000	5	30,000
250,006	4.55	29,997	4.56	29,998
205,475	3.94	29,661	4.02	29,676
171,641	2.47	27,903	3.13	28,179

Table 4 Lifting vehicle with lift and drag coefficients varying with both altitude and velocity

	Exact solution		Series solution	
$\bar{r}$, ft	θ, deg	$\bar{v}$, fps	θ, deg	$\bar{v}$, fps
300,000	5	30,000	5	30,000
250,007	4.55	29,997	4.56	29,998
205,452	3.95	29,661	4.05	29,677
170,447	2.85	27,896	3.74	28,100

The method of systematic, series expansion is applied to the full, two-dimensional, dimensionless, equations of motion. A sequence of linear, first-order differential equations is thus obtained which can be solved immediately by quadrature. The resulting solution is in an analytical form that not only agrees with the exact solution but also serves to estimate the order of accuracy of the existing approximate solutions. As there is practically no restriction imposed in obtaining the solution, its applicability is therefore general within the range of its applicability. Furthermore, as the nondimensional equation is independent of any physical characteristics of a vehicle, its solution has similarity properties.

However, it should be cautioned that the present series solution, being based upon $dv^2/d \log\rho = 0(\epsilon)$ and $d \cos\theta/d \log\rho = 0(\epsilon)$, is not uniformly valid from entry to impact. This is because the order of ρ may change from ϵ or 1 at entry to $1/\epsilon$ at impact. Furthermore, because of the appearance of $\sin\theta$ in the denominator, the present scheme fails when a skip occurs. As a remedy for the former, another series expansion valid for the low altitude can be developed by following the same concept as that for the high altitude. This series expansion for the low altitude, when combined with the present solution for high altitude, may be able to predict the complete trajectory. As a remedy for the latter, an alternate formulation using velocity as independent variable has been adopted, the result of which is reported in a separate paper.[21] The method of series expansion has also been applied to the determination of aerodynamic control for a re-entry vehicle. The details can be found in the forementioned separate paper.[21]

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